Computer Science II

Final Examination

6 May 2009

All questions carry equal marks.

1. Assume that the polynomial

$$p(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n, \quad a_0 \neq 0$$

has a single dominant zero, say r_1. Consider the iteration defined by the following difference equation

$$a_0 x_k + a_1 x_{k-1} + \ldots + a_n x_{k-n} = 0$$

Show that

$$\lim_{k \to \infty} \frac{x_{k+1}}{x_k} = r_1$$

2. Consider the initial value problem

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= -te^{-y}, \quad t \in [0,1] \\ y(0) &= 0 \end{aligned}$$

The Backward Euler scheme to solve this equation is

$$\begin{aligned} y_{n+1} &= y_n - ht_{n+1}e^{-y_{n+1}} =: \phi(y_{n+1}), \quad n = 0, 1, \ldots \\ y_0 &= 0 \end{aligned}$$

The solution y_{n+1} can be obtained by the following fixed point iteration

$$\begin{aligned} y_{n+1}^{(k+1)} &= \phi(y_{n+1}^{(k)}), \quad k = 0, 1, \ldots \\ y_{n+1}^{(0)} &= y_n \end{aligned}$$

Find the condition on h under which the fixed point iterations will converge.

3. An upper Hessenberg matrix $A = [a_{ij}]$, $1 \le i, j \le n$ is such that

$$i > j + 1 \quad \Longrightarrow \quad a_{ij} = 0$$

Write a computer program in C/Fortran/Matlab to perform Gaussian elimination on such a matrix to convert it to upper triangular form. Count the number of additions, subtractions, multiplications and divisions. (NOTE: You only need to discuss the algorithm to obtain the upper triangular form of the matrix.)

4. Given the values $f(-1), f(0), f(+1), f''(-1), f''(0), f''(+1)$, derive the following numerical integration formula

$$\int_{-1}^{+1} f(x)\mathrm{d}x \approx \frac{1}{21}[5f(-1)+32f(0)+5f(+1)]-\frac{1}{315}[f''(-1)-32f''(0)+f''(+1)]$$

which is exact for polynomials of degree ≤ 6

5. Given the N Chebyshev nodes

$$x_i = \cos\left[\frac{(2i+1)\pi}{2N}\right], \quad i = 0, 1, \ldots, N-1$$

the Chebyshev polynomials $T_m(x), T_n(x)$, with $m, n < N$, satisfy the following discrete orthogonality relation

$$\sum_{i=0}^{N-1} T_m(x_i)T_n(x_i) = \begin{cases} 0 & m \ne n \\ N/2 & m = n \ne 0 \\ N & m = n = 0 \end{cases}$$

Given the discrete data $(x_i, f(x_i))$, $i = 0, 1, \ldots, N-1$, use the above orthogonality property to construct the Chebyshev least squares approximation to $f(x)$ using the first $M+1$ Chebyshev polynomials, with $M+1 < N$.

6. Let A be a real $n \times n$ matrix whose eigenvalues are all real, positive and distinct. The solution of $Ax = b$ for some $b \in \mathbb{R}^n$ can be thought of as the steady-state or asymptotic solution of the following system of ordinary differential equations

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= b - Ay, \quad t \ge 0 \\ y(0) &= y^{(0)} \end{aligned}$$

i.e., $x = \lim_{t\to\infty} y(t)$. We can solve the above equation using forward Euler discretization with step τ,

$$\frac{y^{(n+1)} - y^{(n)}}{\tau} = b - Ay^{(n)}$$

Under what condition on τ will the above iterations converge ?

7. Show that there is a unique cubic polynomial $p(x)$ for which

$$\begin{aligned} p(x_0) &= f(x_0) \\ p'(x_1) &= f'(x_1) \\ p''(x_1) &= f''(x_1) \\ p(x_2) &= f(x_2) \end{aligned}$$

where $f(x)$ is a given smooth function and $x_0 \neq x_2$. Derive a formula for $p(x)$.

8. Let A and B be two real $n \times n$ matrices with A being non-singular. Consider solving the linear system

$$\begin{aligned} Az_1 + Bz_2 &= b_1 \\ Bz_1 + Az_2 &= b_2 \end{aligned}$$

where $z_1, z_2, b_1, b_2 \in \mathbb{R}^n$. Find necessary and sufficient conditions for the convergence of the following iterative method

$$\begin{aligned} Az_1^{(m+1)} &= b_1 - Bz_2^{(m)} \\ Az_2^{(m+1)} &= b_2 - Bz_1^{(m)} \end{aligned}$$

for any starting values $z_1^{(0)}$, $z_2^{(0)} \in \mathbb{R}^n$.